An Alternative Technique for Solving Impedance Matching Problems on the Smith Chart

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***Abstract*—this paper presents a numerical method to match an arbitrary complex impedance to a real source impedance using a numerical technique. The motivation of applying the numerical method presented here is to provide a more accurate location of the intersections in comparison with a naked-eye reading on the Smith Chart for different circles (i.e. constant VSWR circle and $1 + jx$ circle).**

***Index Terms*—Matched networking problem, Smith Chart, Transmission line.**

I. Introduction

The ultimate goal of the impedance matching is to minimize the reflection coefficient between a known source output impedance and an arbitrary load impedance connected by a transmission line of a known characteristic impedance, thereby achieving maximum power transfer. This can be accomplished by the means of a low-loss matching network. There are a number of different techniques that can be used to solve the impedance matching problem [2] [3].

One common technique involves the use of the Smith Chart. The Smith Chart is a graphic representation of transmission-line parameters that is commonly used for both numerical calculations and presenting design parameters in a visual setting [1]. The Smith Chart is a four-dimensional (4-D) representation of all possible complex impedances with respect to coordinates defined by the complex reflection coefficient.

The Smith Chart (SC) provides a way to connect the reflection coefficient and the normalized impedance by

$$\Gamma = \frac{\zeta - 1}{\zeta + 1}$$

where ζ is the normalized impedance at $z = \ell$, with ℓ is the distance between the load and the lumped element.

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In general, it is more convenient to use normalized impedances or admittances to design the desired matching network, instead of directly calculating the required reflection coefficient. There are a few common topologies used for impedance matching. Each configuration requires a set of defined parameters. For example, synthesizing a series lumped-element circuit requires the knowledge of the distance between the load and the lumped element in use. In addition, the actual value of the lumped element, such as a capacitor or an inductor, is also needed.

The Smith Chart provides us a convenient and visual way to locate the intersection point, which is used to calculate the actual value of lumped elements. However, it is rather difficult to accurately identify the intersection point under the limit of the naked-eye observation due to the relatively large scale of the grid on a usual Smith Chart. Consequently, a method that can produce accurate values of the intersection points would be advantageous.

This article introduces a numerical method which combines graphical and analytical equations to provide a precise location of the solutions on the Smith Chart. Section II presents an overview of the series lumped-element topology to match an arbitrary load impedance to a purely real source impedance. Section III demonstrates the use of the proposed numerical method to solve a conventional L-section impedance matching problem for an arbitrary load impedance.

II. Series lumped method

The series lumped-element method is usually considered the simplest technique to match an arbitrary complex impedance to a purely real source impedance. The schematic configuration is shown in Figure 1.

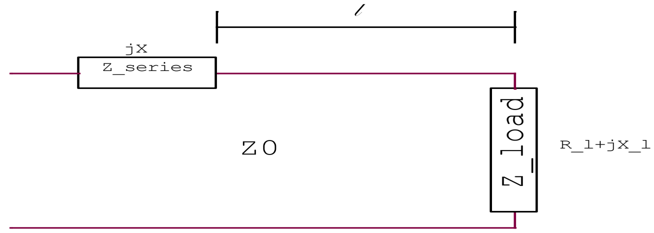

Figure 1: Configuration of lumped-element series matching network

There are two parameters that need to be obtained to design the matching network with this method. The first one is the distance (ℓ) between the load and the lumped element. The second one is the actual value of 'Z_series' which is used to eliminate the reactance of the input impedance. It is assumed that the source impedance is equal to the characteristic impedance of the transmission line Z_0. Using normalized values, we have:

$$\zeta_{in} = \frac{Z_{in}}{Z_0} \quad (1)$$

$$\Gamma_{in} = \frac{\zeta_{in}-1}{\zeta_{in}+1} \quad (2)$$

Under the assumption of a lossless transmission line, the equality $|\Gamma_{load}| = |\Gamma_{input}|$ is satisfied everywhere along the transmission line.

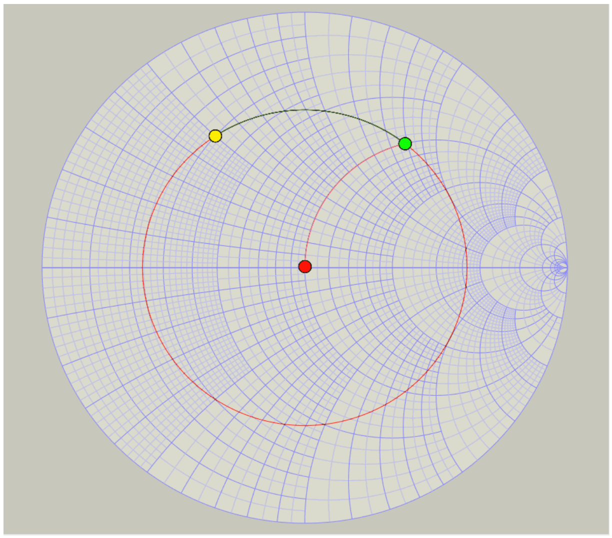

Figure 2: Illustration of the Series-Lumped method on the Smith Chart (The yellow dot represents the normalized load impedance. The green dot represents the intersection of the constant *VSWR* circle with the 1+jx circle. The red dot represents the input impedance after matching (perfect matching)).

Now, the question becomes how to locate the accurate position for the intersection (green dot in Figure 2) of the constant *VSWR* circle and1+jx circle on the impedance Smith Chart.

In order to make the procedure easier to demonstrate, let us consider the following example:

1. Load impedance: $Z_L = 30 + j50\,[\Omega]$
2. Characteristic impedance of transmission line:
   $Z_0 = 100\,[\Omega]$

Then, the step-by-step procedure for finding the numerical value of the intersection (green dot in Figure 2) will be:

(1). Normalize the load impedance.

$$\zeta_L = \frac{30 + j50}{100} = 0.3 + j0.5$$

(2). Find the reflection coefficient (Γ) at the load and construct the constant *VSWR* circle (red circle in Figure 2).

$$\Gamma_L = \frac{\zeta_L - 1}{\zeta_L + 1} = \frac{0.3 + j0.5 - 1}{0.3 + j0.5 + 1} = 0.618\angle 123.425°$$

(3). Before adding the lumped element ,the normalized input impedance should be $\zeta_{in} = 1 + jx$, which lies on the 1+jx circle of the impedance Smith Chart. By applying the equation $\Gamma = \frac{\zeta-1}{\zeta+1}$, the reflection coefficient (Γ_{in}) at the generator (source) terminal is:

$$\Gamma_{in} = \frac{\zeta_{in} - 1}{\zeta_{in} + 1} = \frac{1 + jx - 1}{1 + jx + 1} = \frac{x^2}{4 + x^2} + \frac{2x}{4 + x^2} j$$

(4). Assume the transmission line is lossless; the magnitude of the reflection coefficient should be identical everywhere along this transmission line. The relation $|\Gamma_L| = |\Gamma_{in}|$ will be satisfied (the radius of constant Gamma circle). From step (2) and step (3), the value of x can be calculated as.

$$0.618 = \sqrt{\left(\frac{x^2}{4 + x^2}\right)^2 + \left(\frac{2x}{4 + x^2}\right)^2}$$

$$\Rightarrow \quad x = \pm 1.5722$$

(5). The intersection can be verified by using a compass to draw a constant *VSWR* circle intersecting with 1+jx circle on the impedance Smith Chart. In this case, the value read from the Smith Chart is $\zeta_{in} = 1 \pm j1.5$, which appears very close to the numerical calculation results, but not as accurate as the value obtained by the numerical method.

We can choose either a capacitor or an inductor as the lumped element required to cancel out the reactive part of the original complex input impedance. After adding the lumped element, the input impedance becomes purely real and equal to Z_0.

In general, for a given magnitude of reflection coefficient at the load impedance on a lossless transmission line, the formula of the numerical solution can be generalized as:

$$|\Gamma_{load}| = \sqrt{\left(\frac{x^2}{4 + x^2}\right)^2 + \left(\frac{2x}{4 + x^2}\right)^2} \quad (3)$$

$$\Rightarrow \quad x = \pm \frac{2|\Gamma_{load}|}{\sqrt{1 - |\Gamma_{load}|^2}} \quad (4)$$

$$\Rightarrow \quad \zeta_{in} = 1 \pm j\frac{2|\Gamma_{load}|}{\sqrt{1 - |\Gamma_{load}|^2}} \quad (5)$$

To find a lumped element which has the numerical value $\pm j\frac{2|\Gamma_{Load}|}{\sqrt{1-|\Gamma_{Load}|^2}}$ in a Phasor representation to make the $\zeta_{in} = 1$, we can use the following procedures:

1. Replacing capacitor by the series lumped element.

$$\frac{1}{j\omega C} = -j\frac{2|\Gamma_{load}|}{\sqrt{1 - |\Gamma_{load}|^2}} \quad \rightarrow \quad C = \frac{\sqrt{1 - |\Gamma_{load}|^2}}{2|\Gamma_{load}|} * \frac{1}{\omega}$$

2. Replacing inductor by the series lumped element.

$$j\omega L = j\frac{2|\Gamma_{load}|}{\sqrt{1 - |\Gamma_{load}|^2}} \quad \rightarrow \quad L = \frac{2|\Gamma_{load}|}{\sqrt{1 - |\Gamma_{load}|^2}} * \frac{1}{\omega}$$

This technique can be extended to a shunt lumped-element case by using the admittance Smith Chart instead of its impedance counterpart.

III. L-SECTION METHOD

The L-Section network matching technique is a combination of Series Lumped Method and Shunt Lumped Method. It utilizes purely reactive components such as a capacitor and an inductor to force the input impedance/admittance of an unmatched load to become purely real. The Smith Chart is a powerful tool to design the L-Section network, but first one must locate the required intersection points on the Smith Chart to synthesize an L-Section matching network.

According to the resistance value of the load impedance, there are two types of L-Section networks. Let us assume that the normalized load impedance is $\zeta_{loda} = r + jx$:

❖ Type 1: $r > 1$ (Figure 3)

The key point for synthesizing this Type 1 network is to locate the intersection of $1 + jx$ circle and $y_{load} = g_{load} + jb_{load}$ circle on the admittance Smith Chart, where y_{load} is the normalized load admittance. The steps required to find this point are:

a) Construct the $1 + jx$ circle on the admittance Smith Chart. (red circle in Figure 4)
b) Find the normalized input admittance by locating the intersection of $y_{in} = g_{load} + jb$ and the $1 + jx$ circle. (where g_{load} is fixed, and b is a variable)
c) Convert the input admittance back to impedance and find the proper series reactive component to cancel out the imaginary part of the load impedance.

To further illustrate this technique, let us consider the following example with the procedures illustrated in Figure 4. Assume the given conditions are:

1. Load impedance: $Z_L = 200 - j100\ [\Omega]$
2. Characteristic impedance of transmission line: $Z_0 = 100\ [\Omega]$
3. Operating Frequency: $f = 500\,[MHz]$

Then the procedure for finding the normalized input admittance (Y_1) has two steps:

(1). Normalize the load admittance and construct $1 + jx$ circle on admittance Smith Chart.

$$\zeta_{load} = \frac{200 - j100}{100} = 2 - j \rightarrow y_{load} = \frac{1}{\zeta_{load}} = 0.4 + j0.2$$

(the green cross in Figure 4 represents the normalized load admittance)

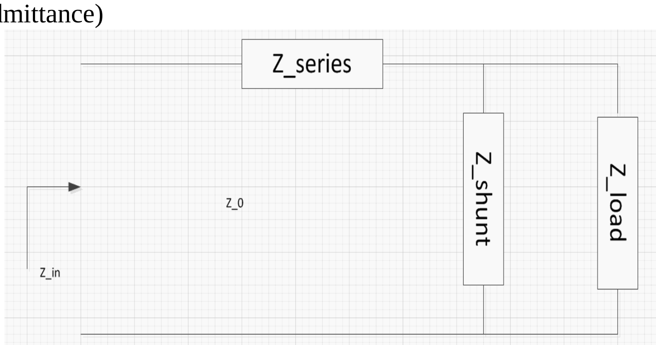

Figure 3: Type 1 L-Section network configuration

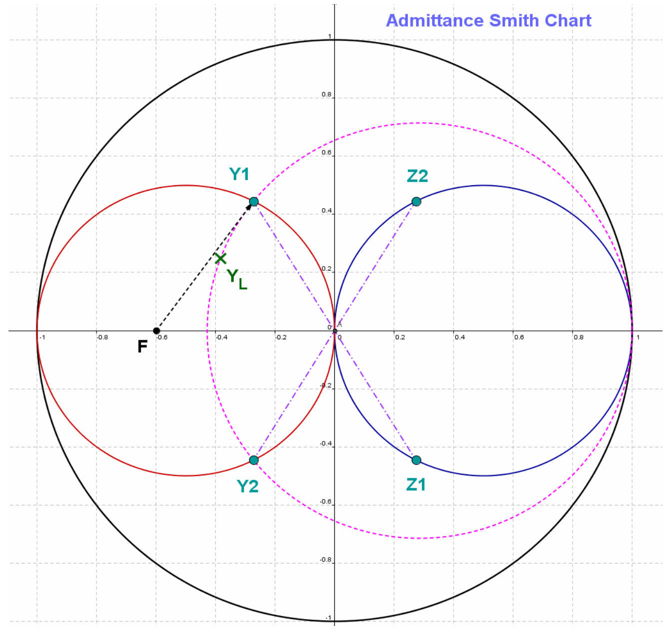

Figure 4: illustration of the procedure to design a Type 1 L-Section matching network. (Red Circle: 1+jx circle on admittance Smith Chart. Blue Circle: 1+jb circle on admittance Smith Chart. Pink Dash-line Circle: $y = g + jb$ circle (g is given). Y_L: normalized load admittance. Y_1 and Y_2: input admittances. Z_1 and Z_2: input impedances.)

(2). Find the input admittances marked as Y_1 and Y_2 in Figure 4 by applying the 'hybrid' numerical method proposed in following.

$$\Gamma_{in} = \frac{\zeta_{in} - 1}{\zeta_{in} + 1} = \frac{1 - y_{in}}{1 + y_{in}} \qquad (6)$$

$$\zeta_{in} = \frac{1}{y_{in}} \qquad (7)$$

y_{in} in Equation (6) is the intersection of $\zeta_{in} = 1 + jx$ circle and $y_{load} = g_{load} + jb_{load}$ circle. Therefore, it is valid to assume that $y_{in} = g_{load} + jb_{in}$ where g_{load} is a fixed value from the normalized load admittance. So far, the only variable for identifying the normalized input admittance is b_{in}. Plugging the $y_{in} = g_{load} + jb_{in}$ into Equation (6), it becomes:

$$\begin{aligned}
\Gamma_{in} &= \frac{1 - y_{in}}{1 + y_{in}} = \frac{1 - g_{load} - jb_{in}}{1 + g_{load} + jb_{in}} \\
&= \frac{(1 - g_{load} - jb_{in})(1 + g_{load} + jb_{in})}{(1 + g_{load} + jb_{in})(1 + g_{load} + jb_{in})} \\
&= \frac{1 - j2b_{in} - b_{in}^2 - g_{load}^2}{(1 + g_{load})^2 + b_{in}^2} \\
&= \frac{1 - b_{in}^2 - g_{load}^2}{(1 + g_{load})^2 + b_{in}^2} + j\frac{2b_{in}}{(1 + g_{load})^2 + b_{in}^2}
\end{aligned}$$

At this point, it is convenient to regard the Smith Chart solely as a complex plane for reflection coefficient (Γ_{in}) which has the range of $y = [-1\ 1]$ and $x = [-1\ 1]$. From a geometric point of view, the $1 + jx$ circle (red circle in Figure 4) has a radius of 0.5 and a center at -0.5, and the normalized input admittance Y_1 lies on this circle. Therefore, the distance between point Y_1 and point F in Figure 4 should be 0.5. The distance between these two points can be obtained by:

$$\left[\frac{1-(b_{in}{}^2+g_{load}{}^2)}{(1+g_{load})^2+b_{in}{}^2}+\frac{1}{2}\right]^2+\left[\frac{2b_{in}}{(1+g_{load})^2+b_{in}{}^2}\right]^2=0.5^2$$

Now, this equation has only one unknown, b_{in}, and it can be solved with numerical techniques, resulting in $b_{in}=\pm 0.4899\cong 0.5$. This result is consistent with the value that can be directly read from Smith Chart.

Once the input admittance is found, the shunt element in Figure 4 can be determined by calculating the difference between the complex part of normalized load admittance and the normalized input admittance (Y_1). The final step is to convert the normalized input admittance to impedance using Equation (7), and then determine the series element by applying the Series Lumped Method.

The two numerical solutions for the shunt element and series element are:

Shunt element:

1. Using a capacitor: $C\approx 0.92\ pF$
2. Using an inductor: $L\approx 46.14\ nH$

Series element:

1. Using an inductor: $L\approx 38.98\ nH$
2. Using a capacitor: $C\approx 2.60\ pF$

*only consider the capacitor-inductor or inductor-capacitor combination.

❖ Type 2: $r<1$ (Figure 5)

For the Type 2 L-Section configuration, a similar design procedure can be used. The basic design idea for the Type 2 case is to use a series lumped element to bring the input impedance (Z_1 in Figure 6) to the $1+jb$ circle by adding a series lumped reactive element on the load impedance (Z_L in Figure 6), and then converting it back to input admittance (Y_1 in Figure 6). The complex part of the input admittance can be cancelled by applying the Shunt-Lumped Method. The input admittance will become a purely real number with the characteristic admittance Y_0.

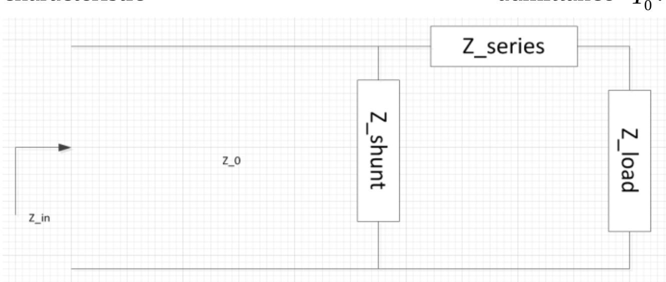

Figure 5: Type 2 L-Section network configuration

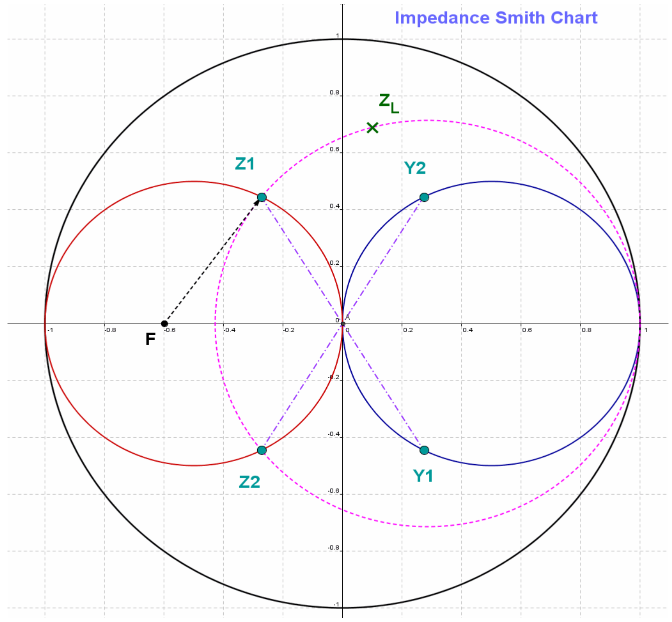

Figure 6: Illustration of the Type 2 L-Section matching network technique. (Red Circle: 1+jb circle on impedance Smith Chart. Blue Circle: 1+jx circle on impedance Smith Chart. Pink Dash-line Circle: $y=r+jx$ circle (r is given). z_L: normalized load impedance. Y_1 and Y_2: input admittance. Z_1 and Z_2: input impedance.)

The key point of designing a Type 2 L-Section matching network relies on locating the normalized input impedance (Z_1 and Z_2 in Figure 6) on the $1+jb$ circle, as the Type 1 L-Section does. The numerical method depicted in the Type 1 network solution is still applicable in this case because the input impedance (Z_1) is on the $\zeta_{in}=r_{in}+jx$ circle where r_{in} is a fixed number, and the only variable is x which is the imaginary part of the input impedance. Once the input impedance is determined, the next step is to use a shunt element to cancel the susceptance of the input admittance.

The flow diagram in Figure 7 illustrates the procedure of L-section impedance matching technique by applying 'hybrid' numerical method.

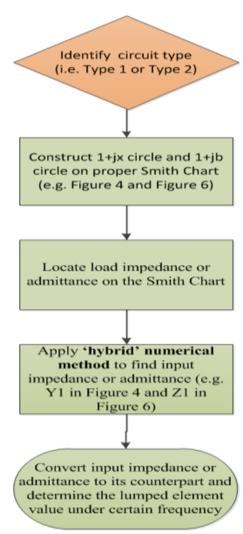

Figure 7: Flow diagram of utilizing 'hybrid' numerical method

IV. COMPARISON WITH OTHER METHODS

There are several numerical methods developed for synthesizing an L-Section matching network. For example, Pozar [2] demonstrated an analytical solution for L-section impedance matching. Using the same example given in Section II and applying Pozar's method, the lumped element values for the shunt and series elements can be obtained:

Shunt element:

1. Using a capacitor: $C \approx 0.92\ pF$
2. Using an inductor: $L \approx 46.14\, nH$

Series element:

1. Using an inductor: $L \approx 38.98\, nH$
2. Using a capacitor: $C \approx 2.60\ pF$

Another purely numerical method has been demonstrated by Rhea [3]. He derived a general formula for the Type 1 L-Section impedance matching network. Rhea's formula should also be applicable to the same example given earlier.

A summary of the results obtained with the three different methods is presented in Table 1.

Table 1 Calculation results with three different methods for $Z_L = 200 - j100\ [\Omega]$**,** $Z_0 = 100\,[\Omega]$**, and** $f = 500\,[MHz]$

	POZER'S	Rhea's	'HYBRID' NUMERICAL
L_1 **(shunt)**	46.14 nH	46.14 nH	46.14 nH
C_1 **(series)**	2.60 pF	2.60 pF	2.60 pF
C_2 **(shunt)**	0.92 pF	0.92 pF	0.92 pF
L_2 **(series)**	38.98 nH	38.98 nH	38.98 nH

Based on the results obtained from Pozar's and Rhea's formula, and comparing them with those obtained from 'hybrid' numerical method outlined here, it is apparent that all three methods agree with each other. Pozar's and Rhea's methods were directly derived from the definition of the impedance matching condition, whereas the 'hybrid' method relies on the Smith Chart and utilizes some geometric properties. This makes the calculation procedure more intuitive than the purely analytical methods. On the other hand, care must be taken when applying these three methods. All three methods are valid under the same assumption that the load impedance is a constant value for all operating frequencies and the transmission line is lossless. In the other words, all three methods are limited to a narrowband network.

However, there are some developed techniques for wideband impedance matching, such as multi-L-section, multi-quarter-wave, etc.

V. SUMMARY

An alternative 'hybrid' numerical method utilizing the geometric knowledge of the Smith Chart is superior than purely numerical methods due to its accuracy of calculation and ease of use. The Smith Chart originated from the representation of reflection coefficient (Γ) on complex coordinate (Cartesian coordinate). Thus, from an algebraic point of view, the intersection point can be determined by introducing the geometric relations in the Cartesian system. Taking advantage of geometric properties on the Smith Chart will give a readily comprehensible calculation. The 'hybrid' numerical method is straightforward to implement and will lead to a much improved precision instead of directly reading the Smith Chart with the naked eye.

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[3] Randy Rhea, "High Frequency Electronics."Vol. 5 No. 3.